

ON MAXIMUM INDUCED MATCHING NUMBERS OF SPECIAL GRIDS

TAYO CHARLES ADEFOKUN¹ AND DEBORAH OLAYIDE AJAYI²

ABSTRACT. A subset M of the edge set of a graph G is an induced matching of G if given any two $e_1, e_2 \in M$, none of the vertices on e_1 is adjacent to any of the vertices on e_2 . Suppose that MIM_G , a positive integer, is the largest possible size of M in G , then, M is the maximum induced matching, MIM , of G and MIM_G is the maximum induced matching number of G . We obtain some upper bounds for the maximum induced matching numbers of some specific grids.

1. INTRODUCTION

For a graph G , let $V(G), E(G)$ be vertex and edge sets respectively and let $e \in E(G)$, we define $e = uv$, where $u, v \in V(G)$. Also, the respective order and size of $V(G)$ and $E(G)$ are $|V(G)|$ and $|E(G)|$. For some $M \subseteq E(G)$, M is an induced matching of G if for all $e_1 = u_i u_j$ and $e_2 = v_i v_j$ in M , $u_k v_l \notin M$, where k and l are from $\{i, j\}$. Induced matching, a variant of the matching problem, was introduced in 1982 by Stockmeyer and Vazirani [8] and has also been studied under the names strong matchings, "risk free" marriage problem. It has found theoretical and practical applications in a lot of areas including network problems and cryptology [3]. For more on induced matching and its applications, see [2], [3], [4], [5] [9].

The size of an induced matching is the number of edges in the induced matching and induced matching M of G with the largest possible size is known as the maximum induced matching which is denoted by MIM , its size, MIM_G , is called the maximum induced matching number (or the strong matching number) of G . Obtaining MIM_G is NP -hard, even for regular bipartite graphs [4]. However, MIM_G of some graphs have been found in polynomial time ([3], [6]).

A grid $G_{n,m}$ results from the Cartesian product of two paths P_n and P_m , resulting in n -rows and m -columns. Marinescu-Ghemaci in [7], obtained the

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MIM_G for $G_{n,m}$, grid where both n, m are even; either of n and m is even and a number of grids $G_{n,m}$ for which nm is odd, which is called the odd grid [1]. Marinescu-Ghemaci also gave useful lower and upper bounds and conjectured that the MIM numbers of grids can be found in polynomial time. Furthermore, by combining the MIM numbers of certain partitions of odd grids, it was shown that for any odd grid $G \equiv G_{n,m}$, $MIM_G \leq \lfloor \frac{nm+1}{4} \rfloor$. This bound was improved on in [1] for the case where $n \geq 9$ and $m \equiv 1 \pmod{4}$. In this paper, the Marinescu-Ghemaci bound for the case where $n \geq 9$ and $m \equiv 3 \pmod{4}$ is considered and more compact values are obtained. The results in this work, combined with some of the results in [7], confirm the MIM numbers of certain graphs, whose MIM numbers' lower bounds were established in [7].

Section 2, of this work, is a review of definitions and preliminary results needed in this work, while in section 3, we obtain the maximum induced matching number of odd grids.

2. DEFINITIONS AND PRELIMINARY RESULTS

Grid, $G_{n,m}$, as defined in this work, is the Cartesian product of paths P_n and P_m with $V(P_n) = \{u_1, u_2, \dots, u_n\}$ and $V(P_m) = \{v_1, v_2, \dots, v_m\}$. We adapt the following notations from [1]: $V_i = \{u_1v_i, u_2v_i, \dots, u_nv_i\} \subset V(G_{n,m}), i \in [1, m]$ and $U_i = \{u_iv_1, u_iv_2, \dots, u_iv_m\} \subset V(G_{n,m}), i \in [1, n]$. For edge set $E(G_{n,m})$ of $G_{n,m}$, if $u_iv_ju_kv_j \in E(G_{n,m})$ and $u_iv_ju_iv_k \in E(G_{n,m})$, we write $u_{(i)}v_j \in E(G_{n,m})$ and $u_iv_{(j)} \in E(G_{n,m})$ respectively.

A saturated vertex v is any vertex on an edge in M , otherwise, v is unsaturated. We define v as saturable if it can be saturated relative to the nearest saturated vertex. Any vertex that is at least distant-2 from any saturated vertex is saturable. The set of all saturated and saturable vertices on a graph G is denoted by $V_{st}G$ and $V_{sb}(G)$ respectively. Clearly, $|V_{st}(G)|$ is even and $V_{st}(G) \subseteq V_{sb}(G)$. Free saturable vertices (FSV) are saturable vertices that can not be on any member of M , $FSV = V_{sb} \setminus V_{st}$. Let G be a $G_{n,m}$ grid, we define $G^{[k]}$ as a $G_{n,k}$ subgrid of G induced by $V_{i+1}, V_{i+2}, \dots, V_{i+k}$.

The following results from [7] on G , a $G_{n,m}$ grid, are useful in this work:

Lemma 2.1. *Let $m, n \geq 2$ be two positive integers and let G be a $G_{n,m}$ grid. Then,*

- (a) *If $m \equiv 2 \pmod{4}$ and n odd then $|V_{sb}(G)| = \frac{mn+2}{2}$; and $|V_{st}(G)| = \frac{mn}{2}$ otherwise;*
- (b) *for $m \geq 3$, m odd, $|V_{sb}(G)| = \frac{nm+1}{2}$, for $n \in \{3, 5\}$.*

Theorem 2.2. *Let G be a $G_{n,m}$ grid with $2 \leq n \leq m$. Then,*

- (a) if n even and m even or odd, then $MIM_G = \lceil \frac{mn}{4} \rceil$;
- (b) if $n \in \{3, 5\}$ then for
 - (i) $m \equiv 1 \pmod{4}$, $MIM_G = \frac{n(m-1)}{4} + 1$
 - (ii) $m \equiv 3 \pmod{4}$, $MIM_G = \frac{n(m-1)+2}{4}$

The following theorem is the statement of the bound given by Marinescu-Ghemaci [7].

Theorem 2.3. *Let G be a $G_{n,m}$ grid, $m, n \geq 2$, mn odd. Then $MIM_G \leq \lfloor \frac{mn+1}{4} \rfloor$.*

3. MAXIMUM INDUCED MATCHING NUMBER OF ODD GRIDS

The following result and the remark describe the importance of the saturation status of certain vertices in $G_{5,p}$ grid, where $p \equiv 2 \pmod{4}$.

Lemma 3.1. *Let G be a $G_{n,m}$ grid and let $\{V_i, V_{i+1}, \dots, V_{i+p}\} \subset G$ induce $G^{[p]}$, a $G_{5,p}$ subgrid of G , where $p \equiv 2 \pmod{4}$. Suppose that M_1 is an induced matching of $G^{[p]}$ and that for $u_3v_i \in V(G^{[p]})$, $u_3v_i \notin V_{st}(G^{[p]})$. Then, $|V_{st}(G^{[p]})| \leq 10k + 4$ and M_1 is not an MIM of $G^{[p]}$.*

Proof. Let $p = 4k + 2$, $G^{[2]}$ and $G^{[p-2]}$ be partitions of G_1 , induced by $\{V_i, V_{i+1}\}$ and $\{V_{i+2}, V_{i+3}, \dots, V_{i+p}\}$, respectively. Since u_3v_i is not saturated in $G^{[2]}$, it easy to check that $|V_{sb}(G^{[2]})| = 5$. From [7], $|V_{sb}(G^{[p-2]})| = |V_{st}(G^{[p-2]})| = 10k$. Thus $|V_{sb}(G^{[p]})| \leq |V_{sb}(G^{[2]})| + |V_{sb}(G^{[p-2]})| \leq 10k + 5$ and therefore, $|V_{st}(G^{[p]})| = 10k + 4$ since $|V_{st}(G)|$ is even, for any graph G . This is a contradiction since by [7] $|V_{st}(G^{[p]})| = 10k + 6$. \square

Remark 3.1. It should be noted that M in Lemma 3.1 will still not be MIM of G if for the vertex set $A = \{u_1v_1, u_5v_1, u_1v_m, u_3v_m, u_5v_m\} \subset V(G)$, any member of A is unsaturated.

Lemma 3.2. *Suppose $u_{\binom{1}{2}}v_i, u_5v_{\binom{i-1}{i}} \in M$ or $u_{\binom{1}{2}}v_i, u_5v_{\binom{i+1}{i+1}} \in M$ where M is an induced matching of G , a $G_{5,m}$ grid, $m \equiv 3 \pmod{4}$, $m \geq 23$ and $1 < i < m$, $i \notin \{4, m-3\}$. Then M is not a MIM of G .*

Proof. Let G be partitioned into $G^{[m(1)]}$ and $G^{[m(2)]}$, which are respectively induced by $A = \{V_1, V_2, \dots, V_i\}$ and $B = \{V_{i+1}, V_{i+2}, \dots, V_m\}$. Suppose that M is an MIM of G .

Case 1: $i \equiv 1 \pmod{4}$. Let $m = 4k + 3$ and set $i = 4t + 1$, where $k \geq 5$ and $t > 0$. Then, $|m(1)| \equiv 1 \pmod{4}$ and $|m(2)| \equiv 2 \pmod{4}$. Since u_1v_i, u_2v_i, u_5v_i and u_5v_{i-1} are saturated vertices in V_i , then the only FSV on V_{i-1} is u_3v_{i-1} .

Suppose that u_3v_{i-1} remains unsaturated. Let $G^{[m(3)]} \subset G^{[m(1)]}$ induced by $\{V_1, V_2, \dots, V_{i-2}\}$, where $|m(3)| \equiv 3 \pmod{4}$. By [7], $|V_{st}(G^{[m(3)]})| = 10t - 4$. Thus, $|V_{st}(G^{[m(1)]})| \leq 10t$. Suppose that u_3v_{i-1} is saturated, then, $u_5v_{(i-2)} \in M$. Thus, $u_3v_{i-3} \in V_{i-3} \subset G^{[m(4)]}$, unsaturable, where $G^{[m(4)]}$ is $G^{[m(3)]} \setminus V_{i-2}$. Note that $|m(4)| \equiv 2 \pmod{4}$. From Lemma 3.1, therefore, $|V_{st}(G^{[m(4)]})| \leq 10t - 6$ and thus, $|V_{st}G^{[m(1)]}| \leq 10t - 6 + 6 = 10t$. Now, since u_1v_i, u_2v_i and u_5v_i are saturated vertices in V_i , then, $u_3v_{i+1}, u_4v_{i+1} \in V(G^{[m(2)]})$ are saturable vertices in $G^{[m(2)]}$.

Claim: Edge $u_{(3)}v_{i+1}$ belongs to M .

Reason: Suppose that both u_3v_{i+1} and u_4v_{i+1} are not saturated, then V_{i+1} contains no saturable vertices. Let $G^{[m(2)]} \setminus \{V_{i+1}\} = G^{[m(5)]}$, where $|m(5)| \equiv 1 \pmod{4}$. Thus, $|V_{st}(G)| \leq |V_{st}G^{[m(1)]}| + |V_{st}(G^{[m(5)]})| = 10k + 2$. This implies that M requires at least four more saturated vertices to be *MIM* of G . However, $|V_{sb}(G^{[m(5)]})| = 10(k - t) + 3$ and suppose $u_3v_{i+1}, u_4v_{i+1} \in V_{st}(G)$, then $|V_{st}(G)| \leq 10k + 5$, which in fact, is $|V_{st}(G)| = 10k + 4$. Thus if $u_{(2)}v_i, u_5v_{(i-1)} \in M$, then M is not an *MIM* of G .

Suppose that $u_{(2)}v_i, u_5v_{(i-1)} \in M$. Let $G^{[n(1)]} = G^{[m(1)]} \setminus \{V_i\}$ and $G^{[n(2)]} = G^{[m(2)]} + \{V_i\}$. Now, $|n(1)| \equiv 0 \pmod{4}$ and $|n(2)| \equiv 3 \pmod{4}$. We can see that $|V_{st}(G^{[n(2)]})| = 10(k - t) + 6$. Now, on $V_{i-1} \subset G^{[n(1)]}$, only vertices u_3v_{i-1} and u_4v_{i-1} are saturable. Suppose they are both not saturated after all. Let $G^{[n(3)]} \subset G^{[n(1)]}$ be induced by $\{V_1, V_2, \dots, V_{i-2}\}$, where $|n(3)| \equiv 3 \pmod{4}$. $|V_{st}(G^{[n(3)]})| = 10t - 4$. Thus $|V_{st}(G)| = 10k + 2$. Therefore, M requires four saturated vertices to be *MIM* of G . Now, $|V_{sb}(G^{[n(3)]})| = 10t - 2$, and thus, $V(G^{[n(3)]})$ contains two extra *FSV*, say, v_1, v_2 which are not adjacent. Thus, the maximum number of saturable vertices from the vertex set $v_1, v_2, u_3v_{i-1}, u_4v_{i-1}$ is 2. Therefore, $|V_{st}(G)| \leq 10k + 4$, which is a contradiction.

Case 2. For $i \equiv 2 \pmod{4}$. Let $G^{[p(1)]}$ and $G^{[p(2)]}$ be partitions of G induced by $\{V_1, V_2, \dots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \dots, V_m\}$, with $m = 4k + 3$ and $i = 4t + 2$. Let $u_{(2)}v_i$ and $u_5v_{(i-1)} \in M$. Since $u_{(2)}v_i$ belongs M of G , then u_3v_i cannot be saturated. Thus, $|V_{st}(G^{[p(2)]})| \geq 10(k - t) + 2$ for M to be maximal. It can be seen that $|p(2)| \equiv 1 \pmod{4}$. Now, u_3v_{i+1} and u_4v_{i+1} are saturable vertices in V_{i+1} . Suppose both of them are not saturated, then for $G^{[p(3)]}$ induced by $\{V_{i+2}, V_{i+3}, \dots, V_m\}$, where $|p(3)| \equiv 0 \pmod{4}$, $|V_{st}(G^{[p(3)]})| \leq 10(k - t)$. Thus u_3v_{i+1} and u_4v_{i+1} are saturable vertices and in fact, $u_{(3)}v_{i+1} \in M$. On V_{i+2} , therefore, there exists three saturable vertices u_1v_{i+1}, u_2v_{i+2} and u_5v_{i+5} . Suppose none of these three vertices are saturated. Then, $|V_{st}(G^{[p(3)]})| \leq |V_{st}(G^{[p(4)]})| + 2$, with $G^{[p(4)]}$ induced by $\{V_{i+3}, \dots, V_m\}$ and $|p(4)| \equiv 3 \pmod{4}$.

and thus, $|V_{st}(G^{[p(2)]})| \leq 10(t-k) - 2$. Therefore it requires extra four saturated vertices for M to be maximal. There exist two other saturable vertices, $v_1, v_2 \in V(G^{[p(4)]})$ (since $V_{st}(G^{[p(4)]}) = 10(k-t) - 4$ and $V_{sb}(G^{[p(4)]}) = 10(k-t) - 2$). Clearly, v_1, v_2 are not adjacent, else they would have formed an edge in M . Suppose $v_1, v_2 \in V_{i+3}$. For v_1 and v_2 to be saturated, they have to be u_5v_{i+3} and one of u_1v_{i+3} and u_2v_{i+3} . Thus, $u_5v_{i+3} \in M$ and one of u_1v_{i+3} or u_2v_{i+3} belongs to M . Let $G^{[p(5)]}$ be induced by $\{V_{i+4}, \dots, V_m\}$, where $|p(5)| \equiv 2 \pmod{4}$. Now, since $u_5v_{i+3} \in M$, then $u_5v_{i+5} \in V_{i+4}$ is unsaturable and therefore, by Remark 3.1, $|V_{st}(G^{[p(5)]})| = 10(k-t-1) + 4$ and thus, $|V_{st}(G^{[p(2)]})| = 10(k-t)$, which is less than required.

The case of $u_5v_{i+1} \in M$ is the same as the case of $u_5v_{i-1} \in M$ for $i \equiv 2 \pmod{4}$.

Case 3: $i \equiv 0 \pmod{4}$, $i \geq 6$ or $i \leq m-5$, with $u_{(1)}v_i, u_5v_{i-1} \in M$. Let $G^{[r(1)]}$ and $G^{[r(2)]}$ be partitions of G which are induced respectively by $\{V_1, V_2, \dots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \dots, V_m\}$. Since $i \equiv 0 \pmod{4}$, then $|r(1)| \equiv 0 \pmod{4}$, while $|r(2)| \equiv 3 \pmod{4}$. Also, $u_5v_{i-1} \in M$, implies u_5v_{i-1} is unsaturable. Since $i-2 \equiv 2 \pmod{4}$, then by Lemma 3.1 and Remark 3.1, $|V_{st}(G^{[r(1)]})| \leq 10t - 2$, implying that for M to be maximal, $|V_{st}(G^{[r(2)]})| \geq 10(k-t) + 8$. It can be seen that V_{i+1} has two only saturable vertices u_3v_{i+1}, u_4v_{i+2} left. It should also be noted that if any of u_3v_{i+1} and u_4v_{i+2} is saturated, then u_3v_{i+3} can not be saturated in $G^{[r(3)]}$, a subgrid of $G^{[r(2)]}$ induced by $\{V_{i+2}, V_{i+3}, \dots, V_m\}$, with $|r(3)| \equiv 2 \pmod{4}$. Thus suppose $u_3v_{i+1}, u_4v_{i+2} \in V_{st}(G)$, then $|V_{st}(G)| \leq 10(k-t) + 4$. Likewise, if $u_3v_{i+1}, u_4v_{i+2} \notin V_{st}(G)$, $|V_{st}(G)| \leq 10t - 2 + 10(k-t) + 6$.

The case of $u_5v_{i+1} \in M$ follows the same argument as the case of $u_5v_{i-1} \in M$. \square

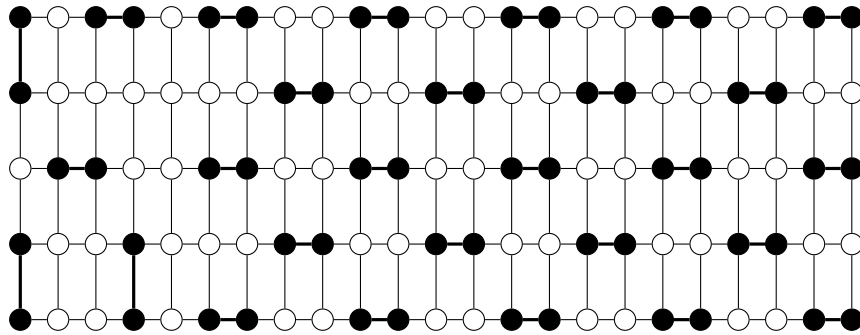


Figure 1. A Grid $G \cong G_{5,23}$ with $MIM_G = 28$, $u_1v_1, u_1v_4 \in MIM$ of G

- Remark 3.2.** (a) In the case of $i \equiv 0 \pmod{m}$ in 3.2, M remains a MIM when $i = 4$ or when $i = m - 3$ as seen in Figure 1 of $|MIM| = 28$ of $G_{5,23}$.
- (b) It should be noted that the case of $i \equiv 3 \pmod{4}$ has been taken care of by the case of $i \equiv 1 \pmod{4}$ by 'flipping' the grid from right to left or vice versa.
- (c) From Lemma 3.2, we see that if for some induced matching M of $G_{5,m}$, $m \equiv 3 \pmod{4}$, $u_{(\frac{1}{2})}v_i$ and $u_5v_{(\frac{i}{2})}$ (or $u_5v_{(\frac{i+2}{2})}$) $\in M$, then M is not a maximal induced matching of G for any $1 < i < m$.

Next we investigate some M of $G_{5,m}$ if it contains $u_{(\frac{1}{2})}v_i$ and u_5v_i .

Lemma 3.3. Suppose $G = G_{5,m}$, where $m \geq 23$ and $m \equiv 3 \pmod{4}$. Let $u_{(\frac{1}{2})}v_i, u_{(\frac{4}{5})}v_i \in M$, an induced matching of G and $1 < i < m$, $i \not\equiv 0 \pmod{4}$ then M is not a MIM of G .

Proof. Suppose that $i \equiv 2 \pmod{4}$. Let $G^{[m(1)]}$ and $G^{[m(2)]}$ be partitions of G induced by $\{V_1, V_2, \dots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \dots, V_m\}$. Since $u_{(\frac{1}{2})}v_i, u_{(\frac{4}{5})}v_i \in M$, then, u_3v_i is unsaturated. Let $i = 4t + 2$, for some positive integer t , by Lemma 3.2 then $|V_{st}(G^{[m(1)]})| = 10t + 4$. Now, only u_3v_{i+1} is saturable on V_{i+1} . Let $G^{[m(3)]} \subset G^{[m(2)]}$, induced by $\{V_{i+2}, \dots, V_m\}$. Clearly $|m(3)| = |m(2)| - 1 = 4(k - t)$. Therefore, $|V_{st}(G^{[m(3)]} + u_3v_i)| \leq 10(k - t) + 1$, which in fact is $10(k - t)$. Thus, $|V_{st}(G)| = 10k + 4$.

Now, suppose $i \equiv 1 \pmod{4}$. Let $G^{[n(1)]}$ be induced by $\{V_1, V_2, \dots, V_i\}$ and let $G^{[n(2)]}$ be induced by $\{V_{i+1}, V_{i+2}, \dots, V_m\}$. Since $|n(1)| = 4t + 1$, it is easy to see that $|n(2)| \equiv 2 \pmod{4}$ and hence, $|n(2)| = 4(k - t) + 2$.

Claim. For M to be maximal, both u_3v_{i-1} and u_3v_{i+1} must be saturated.

Reason: Suppose, say u_3v_{i-1} is not saturated. Then, no vertex on V_{i-1} is saturable. Now, let $\{V_1, V_2, \dots, V_{i-2}\}$ induce grid $G^{[n(3)]}$, with $|n(3)| \equiv 3 \pmod{4}$. $|V_{st}(G^{[n(3)]})| = 10t - 4$, and thus, $|G^{[n(1)]}| = 10t$. Also, Let $G^{[n(4)]}$ be induced by $\{V_{i+2}, V_{i+3}, \dots, V_m\}$. Since $|n(4)| = 4(k - t) + 1$, then for $G^{[n(4)]} + u_5v_{i+1}$, $|V_{sb}[(G^{[n(4)]}) + u_3v_{i+1}]| = 10(k - t) + 4$. Therefore, $|V_{st}(G)| \leq 10k + 4$. Now Suppose $u_3v_{(\frac{i-2}{2})} \in M$. Then, given $G^{[n(5)]}$, induced by $\{V_1, V_2, \dots, V_{i-3}\}$. We can see that $|n(5)| \equiv 2 \pmod{4}$. By Lemma 3.1, $|V_{st}(G^{[n(5)]})| = 10t - 6$. Thus, $|V_{st}(G^{[n(1)]})| = 10t$ and therefore, $|V_{st}(G)| \leq 10k + 4$. \square

Remark 3.3. Like in Remark 3.2, for $i \equiv 0 \pmod{4}$, it can be seen that $u_{(\frac{1}{2})}v_1, u_{(\frac{1}{2})}v_4$ or $u_{(\frac{1}{2})}v_{m-3}, u_{(\frac{1}{2})}v_m$ can be in M if M is MIM of G . Also given $i \equiv 0 \pmod{4}$ and $4 < i < m - 3$, for at most only one i , from 1 to m , $u_{(\frac{1}{2})}v_i$ can be a member of maximal M .

Next we investigate the maximality of the induced matching of $G = G_{5,m}$, $m \equiv 3 \pmod{4}$.

Lemma 3.4. *Let $u_{(\frac{1}{2})}v_i, u_4v_{(i-1)} \in M$ or $u_{(\frac{1}{2})}v_i, u_4v_{(i+1)} \in M$ where M is an induced matching of G , a $G_{5,m}$ grid, $m \equiv 3 \pmod{4}$, $m \geq 23$ and $1 < i < m$, $i \not\equiv 0 \pmod{4}$. Then M is not a MIM of G .*

Proof. Case 1: Let $i \equiv 1 \pmod{4}$. Suppose that $m = 4k + 3$ and $i = 4t + 1$, $t \geq 1$. Let $G^{[m(1)]}$ and $G^{[m(2)]}$ be two partitions of G , induced by $\{V_1, V_2, \dots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \dots, V_m\}$ respectively. Since $u_{(\frac{1}{2})}v_i, u_4v_{(i-1)} \in M$, then there is no other saturated vertex on both of V_{i-1} and V_i . Let $G^{[m(3)]} \subset G^{[m(1)]}$ be a grid induced by $\{V_1, V_2, \dots, V_{i-2}\}$. Now, $n(3) \equiv 3 \pmod{4}$. Therefore, $|V_{st}(G^{[m(3)]})| = 10t - 4$ and hence, $|V_{st}(G^{[m(1)]})| = 10t$. Now, $|m(2)| \equiv 2 \pmod{4}$, since $u_{(\frac{1}{2})}v_i \in M$, then $u_1v_{i+1} \in V_{i+1}$ is unsaturable. From a previous result, $|V_{st}(G^{[n(2)]})| = 10(k - t) + 4$ and thus, $|V_{st}(G)| = 10k + 4$. For $u_4v_{(i+1)} \in M$. Let $G^{[n(1)]}$ and $G^{[n(2)]}$ be induced by $G^{[m(1)]} \setminus V_i$ and $G^{[m(2)]} + V_i$. Then, $|n(1)| \equiv 0 \pmod{4}$ and $|n(2)| = 4(k - t) + 3$. It can be seen that on V_{i-1} , only u_3v_{i-1} and u_5v_{i-1} are saturable vertices.

Claim: Vertices u_3v_{i-1} and u_5v_{i-1} are not saturable for M to be maximal.

Reason: Suppose without loss of generality, that any of u_3v_{i-1} and u_5v_{i-1} is saturated, say u_5v_{i-1} . Then $u_5v_{(i-2)} \in M$. This implies that v_5v_{i-3} is not saturable in V_{i-3} . Now $\{V_1, V_2, \dots, V_{i-3}\}$ induces a grid $G^{[n(4)]}$ and $|n(4)| \equiv 2 \pmod{4}$. Then, $|V_{st}(G^{[m(4)]})| = 10t - 6$ and thus, $|V_{st}(G^{[n(1)]})| = 10t - 4$. Now, since $|n(2)| = 4(k - t) + 3$, $|V_{st}(G^{[m(2)]})| = 10(k - t) + 6$ and therefore, $|V_{st}(G)| = 10k + 2$. Case 2: For $i \equiv 2 \pmod{4}$. Let $G^{[n(1)]}$ and $G^{[n(2)]}$ be two partitions of G , induced by $\{V_1, V_2, \dots, V_i\}$ and $\{V_{i+1}, V_{i+2}, \dots, V_m\}$ respectively. Since $u_{(\frac{1}{2})}v_i$ and $u_4v_{(i-1)} \in M$, vertex $u_5v_i \in V_{sb}(G^{[n(1)]})$, and therefore, $|V_{st}G^{[n(1)]}| = 10t + 4$, where $|n(1)| = 4t + 2$. Also, only u_3v_{i+1} and u_5v_{i+1} are saturable on V_{i+1} . Suppose without loss of generality, that both u_3v_{i+1} and u_5v_{i+1} are saturated and thus, $u_3v_{(i+2)}, u_5v_{(i+2)} \in M$. Now, suppose that $G^{[n(4)]}$ is induced by $\{V_{i+3}, V_{i+4}, \dots, V_m\}$, with $|n(4)| = 4(k - t - 1) + 3$. By following the techniques employed earlier, it can be shown that $|V_{st}(G)| \leq |V_{st}(G^{[n(1)]})| + |V_{st}(G^{[n(2)]})| \leq 10k + 4$. The $u_4v_{(i+4)}$ case, has the same proof as the $u_4v_{(i-1)}$ case. \square

Remark 3.4. There can be only one edge $u_{(\frac{1}{2})}v_i \in M$ for which M is MIM of $G_{5,m}$, if M contains $u_{(\frac{1}{2})}v_i$ and $u_4v_{(i-1)}$ (or $u_4v_{(i+1)}$), and in this case, $i \equiv 0 \pmod{4}$ as shown in Figure 2.

Remark 3.5. It should be noted that the proof of the $i \equiv 1 \pmod{4}$ in Lemma 3.4 will hold for $i \equiv 3 \pmod{4}$ by flipping the grid from right to left.

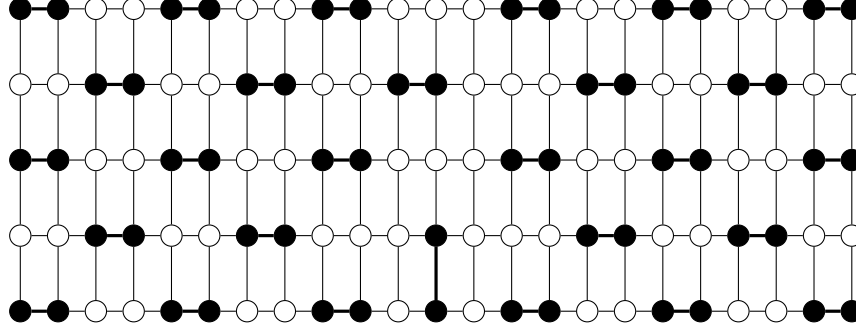


Figure 2. A $G \equiv G_{5,23}$ Grid with $MIM_G = 28$, $u_{\frac{1}{2}}v_i, i \equiv 0 \pmod{4}$

The previous results and remarks yield the following conclusion.

Corollary 3.5. *Suppose that $m \geq 23$ and M is the MIM of G , some $G_{5,m}$ grid. Then, if for at most some positive integer i , $1 < i < m$, $u_{(\frac{1}{2})}v_i \in M$, then, $i \equiv 0 \pmod{4}$.*

Lemma 3.6. *Let M be a matching of $G_{5,m}$ with $m \equiv 3 \pmod{4}$ and let $u_{(\frac{1}{2})}v_i, u_{(\frac{1}{2})}v_j \in M$, $1 < i < j < m$, such that $i \equiv 0 \pmod{4}$ and $j \equiv 0 \pmod{4}$, then M is not an MIM of G .*

The claim in Lemma 3.6 can easily be proved using earlier techniques and Lemma 3.1 and Remark 3.1.

Remark 3.6. It should be noted from the previous results and from Corollary 3.5 that if M is the MIM of $G_{5,m}$, $m \equiv 3 \pmod{4}$, then at most, M contains two edges of the form $u_{(\frac{1}{2})}v_i, u_{(\frac{1}{2})}v_j$ and j can only be 4 when $i = 1$ or i can only be $m - 3$ when $j = m$.

Theorem 3.7. *Let M be the MIM of G , a $G_{5,m}$ grid and let M contain $u_{(\frac{1}{2})}v_1$ and $u_{(\frac{1}{2})}v_4$ (or $u_{(\frac{1}{2})}v_{m-3}$ and $u_{(\frac{1}{2})}v_m$). Then there are at least $2k + 2$ saturated vertices on $U_1 \subset G$.*

Proof. For $u_{(\frac{1}{2})}v_1$ and $u_{(\frac{1}{2})}v_4$ to be in M , either $u_{(\frac{4}{5})}v_4 \in M$ or $u_5v_{(\frac{3}{4})} \in M$. Now, let $\{V_6, V_7, \dots, V_m\}$ induce $G^{[m(1)]} \subset G$. Clearly, $|m(1)| \equiv 2 \pmod{4}$ and $|V_{st}(G^{[m(1)]})| = 10k - 4$. Let $G^{[m(1)]} \setminus \{u_1v_6, u_1v_7, \dots, u_1v_m\}$ induce $G^{[m(2)]} \subset G^{[m(1)]}$. Then, $G^{[m(2)]}$ is a $G_{4,m-5}$ subgrid of $G^{[m(1)]}$. Now, $|V_{st}(G^{[m(2)]})| \leq 8k - 4$. Thus for $V(U_1) \subset V(G^{[m(1)]})$, $|V(U)| \geq 2k$. Thus, U_1 contains at least $2k + 2$ (i.e., $\frac{m-1}{2}$) saturated vertices. \square

Next we investigate $G_{3,m}$, where $m \equiv 3 \pmod{4}$.

Lemma 3.8. *Suppose that G is a $G_{3,m}$ grid with $m \equiv 3 \pmod{4}$ and M is an induced matching of $G_{3,m}$, with $\left\{u_{(\frac{1}{2})}v_i, u_{(\frac{1}{2})}v_{i+2}, u_{(\frac{1}{2})}v_j, u_{(\frac{1}{2})}v_{j+2}\right\} \in M$ and $i+2 \geq j$. Then M is not a MIM of G ,*

Proof. Suppose $i+2 \geq j$. Since $m = 4k+3$, $|V_{sb}(G)| = 6k+5$ and $|V_{st}(G)| = 6k+4$. Thus, G contain at most one FSV . Now from the conditions in the hypothesis, it is clear that u_3v_{i+1} and u_3v_{j+1} are $FSVs$ in G , which is a contradiction. Same argument hold if $i+2 = j$ since both u_3v_{i+1} and u_3v_{i+3} are $FSVs$ in G . \square

Remark 3.7. Suppose that G_n is $G_{3,n}$, a subgrid of $G_{3,m}$ and induced by $\{V_{i+1}, V_{i+2}, \dots, V_{i+n}\}$ and G' is a subgraph of G , with $G' = G_n + \{u_3v_i, u_3v_{i+n+1}\}$, then the following are easy to verify. For

- (a) $n \equiv 0 \pmod{4}$, $|V_{st}(G')| \leq |V_{sb}(G_n)| + 2$
- (b) $n \equiv 1 \pmod{4}$, $|V_{st}(G')| \leq |V_{sb}(G_n)| + 2$
- (c) $n \equiv 2 \pmod{4}$, $|V_{st}(G')| = |V_{sb}(G_n)|$
- (d) $n \equiv 3 \pmod{4}$, $|V_{st}(G')| \leq |V_{sb}(G_n)| + 1$

Lemma 3.9. *Let $u_{(\frac{1}{2})}v_j, u_{(\frac{1}{2})}v_{j+3}, u_{(\frac{1}{2})}v_k, u_{(\frac{1}{2})}v_{k+3}, u_{(\frac{1}{2})}v_l, u_{(\frac{1}{2})}v_{l+3}$ be in M an induced matching of G a $G_{3,m}$ grid and $m \equiv 3 \pmod{4}$. Then M is not MIM of G .*

Proof. Case 1: Let $m = 4p+3$, $j+3 = k$ and $l = k+3$. Suppose $G^{|m(1)|}$ is a subgraph of G , induced by $\{V_{j-1}, V_j, \dots, V_{i+4}\}$. Then $|m(1)| = 12$, with u_3v_{j-1} and u_3v_{i+4} as $FSVs$. For one of u_3v_{j-1} and u_3v_{i+4} to be relevant for M to be MIM of G , say u_3v_{j-1} , then for $G^{|m(2)|}$, induced by $\{V_1, V_2, \dots, V_{j-2}\}$, $|V_{sb}(G^{|m(2)|})|$ must be odd, which can only be if $j-2 \equiv 3 \pmod{4}$. So, suppose $j-2 \equiv 3 \pmod{4}$, then $|V_{st}(G^{|m(2)|}) + u_3v_{j-1}| \leq |V_{sb}(G^{|m(2)|})| + 1 = 6q+6$, where $|m(2)| = 4q+3$, for $q \geq 1$, since $|m(1)| = 12$ and $|n(2)| \equiv 3 \pmod{4}$. Now let $G^{|m(3)|} = G^{|m(1)|} \cup G^{|m(2)|}$, where $|m(3)| = |m(1)| + |m(2)| \equiv 3 \pmod{4}$ and $G^{|m(4)|} \subset G$ be defined as a subgrid of G induced by $\{V_{i+5}, V_{i+6}, \dots, V_m\}$. Clearly, $|m(4)| \equiv 0 \pmod{4}$. Since $|V_{sb}(G^{|m(4)|})| = |V_{st}(G^{|m(4)|})|$, which is even, then $|V_{st}(G^{|m(4)|} + u_3v_{i+4})| = |V_{st}(G^{|m(4)|})| = 6p-6q-18$. Now, it can be seen that $|V_{st}(G^{|m(1)|}) \setminus \{u_3v_{j-1}, u_3v_{l+4}\}| = 14$. Therefore, $|V_{st}(G)| \leq 6p+2$ instead of $6p+4$, and hence a contradiction.

Case 2: Suppose that $j+3 < k$ and $k+3 < l$. As in Case 1 and without loss of generality, let $j-2 \equiv 3 \pmod{4}$ and let $G^{|m(2)|}$ still be induced by $\{V_1, V_2, \dots, V_{j-2}\}$. Also, let $G^{|m(4)|}$ be induced by $\{V_{l+5}, V_{l+6}, \dots, V_m\}$, and set $|m(4)| \equiv 3 \pmod{4}$. Thus, u_3v_{j-1} and u_3v_{i+4} are both relevant for M to be a MIM of G and $|V_{st}(G^{|m(2)|} + V_{j-1})| = |V_{sb}(G^{|m(2)|})| + 1$ and $|V_{st}(G^{|m(4)|} +$

$|V_{l+4}| = |V_{sb}(G^{[m(4)]})| + 1$. Set $G^{[m(2)]} + V_{j-1} = G^{[m(2^+)]}$ and set $G^{[m(4)]} + V_{i+4} = G^{[m(4^+)]}$ and let $\{V_j, V_{j+1}, V_{j+2}, V_{j+3}\}$ induce $G^{[m(5)]}$ while $\{V_i, V_{i+1}, V_{i+2}, V_{i+3}\}$ induces $G^{[m(6)]}$. Furthermore, let $G^{[m(5^+)]} = G^{[m(5)]} + V_{j+4}$ and $G^{[m(6^+)]}$ contain, say, h columns of V_i in all, where $h \equiv 2 \pmod{4}$. Therefore, for $G^{[m(7)]} = G \setminus \left\{ G^{[m(2^+)]} \cup G^{[m(4^+)]} \cup G^{[m(5^+)]} \cup G^{[m(6^+)]} \right\}$, $|m(7)| = m - h = b \equiv 1 \pmod{4}$.

Let $b = 4a + 1$, for some positive integer a and let $G^{[m(4)]} \subset G^{[m(7)]}$, where $G^{[m(7)]}$ is induced by $\{V_k, V_{k+1}, V_{k+2}, V_{k+3}\}$. Certainly, $u_3v_{k-1}, u_3v_{k+4}, u_3v_{j+4}, u_3v_{l-1} \in V_{sb}(G)$. Now, let $G^{[m(4)]}$ be induced by $\{V_k, V_{k+1}, V_{k+2}, V_{k+3}\}$ and $G^{[m(4^+)]}$ be induced by $G^{[m(4)]} + \{V_{k-1}, V_{k+4}\}$, with $|4 + +| = 6$. So, $b - 6 \equiv 3 \pmod{4}$, which is odd and thus can only be the sum of an even and an odd positive integer. Therefore, let $G^{[m(8)]}$ and $G^{[m(9)]}$ be induced by $\{V_{j+5}, V_{j+6}, \dots, V_{k-2}\}$ and $\{V_{j+5}, V_{j+6}, \dots, V_{l-2}\}$, with $|m(8)| + |m(9)| = b$. Suppose thus, that $|m(8)| \equiv 0 \pmod{4}$, then, $|m(9)| \equiv 3 \pmod{4}$ and suppose $|m(8)| \equiv 1 \pmod{4}$, then $|m(9)| \equiv 2 \pmod{4}$. For $|m(8)| \equiv 0 \pmod{4}$, let $G^{[m(10)]} = G^{[m(2^+)] + |m(5^+)]}$ be $G^{[m(2^+)]} \cup G^{[m(5^+)]}$ and $G^{[m(11)]} = G^{[m(6^+)] + |m(4^+)]}$ be $G^{[m(6^+)]} \cup G^{[m(4^+)]}$, where $|m(2^+)| + |m(5^+)| = 4q + 9$ and $|m(4^+)| + |m(6^+)| = 4r + 9$, where $|m(4)| = 4r + 3$. Therefore, as defined, $b = |m(7)| = 4p - 4q - 4r - 15$ and thus $b - 6 = 4(p - q - r - 6) + 3$. Set $p - q - r - 6 = f$. Now, for $|m(8)|$ and $|m(9)|$, if $|m(8)| = 4g$, for some positive integer g , then $|m(9)| = 4(f - g) + 3$. Next we sum the maximal values of the subgrid of G as follows: $|V_{st}(G)| \leq |V_{st}(G^{[m(2^+)]} \cup G^{[m(5^+)]})| + |V_{st}(G^{[m(8)]} + \{u_3v_{j+4}, u_3v_{k-1}\})| + |V_{st}(G^{[m(4)]})| + |V_{st}(G^{[m(9)]} + \{u_3v_{k+4}, u_3v_{l-1}\})| + |V_{st}(G^{[m(6)]} \cup G^{[m(4^+)]})| \leq 6p + 2$, which is less than $6p + 4$ and hence a contradiction. For $|m(8)| \equiv 1 \pmod{4}$, and $|m(9)| \equiv 2 \pmod{4}$, we have $|m(8)| = 4g + 1$ and hence $|m(9)| = 4(f - g) + 2$ and $|V_{st}(G^{[m(9)]} + \{u_3v_{k+4}, u_3v_{l-1}\})| = 6(f - g) + 4$ and thus, $|V_{st}(G)| \leq 6p + 2$.

Case 3: Suppose $j + 3 = k$ or $k + 3 = i$. Without loss of generality, let $j + 3 = k$. Suppose as in Case 2, $j - 2 \equiv 3 \pmod{4}$ and $m - (i + 4) \equiv 3 \pmod{4}$. Let $G^{[n(1)]} \subset G$, a $G_{3,9}$ subgrid of G be induced by $\{V_{j-1}, v_j, \dots, V_{j+7}\}$. Then for $G^{[n(2)]} = G^{[m(2)]} \cup G^{[n(1)]}$, $|n(2)| = |m(2)| + |n(1)|$, $|n(2)| \equiv 0 \pmod{4}$. Likewise, suppose $\{V_{i-1}, V_i, \dots, V_m\}$ induces $G^{[n(3)]}$, for which $|n(3)| \equiv 1 \pmod{4}$. If $|n(2)|$ and $|n(3)|$ are $4g$ and $4r + 1$ respectively, then $|n(4)| \equiv 2 \pmod{4}$. So far, $G^{[n(4)]}$ is induced by $\{V_{i+8}, V_{i+9}, \dots, V_{l-2}\}$ and by Remark 3.7, $|V_{st}(G^{[n(4)]})| + |\{u_3v_{j+7}, u_3v_{l-1}\}| = |V_{sb}(G^{[n(4)]})|$. By a summation similar to the one at the end of case 2, $|V_{st}(G)| \leq |V_{st}G^{[n(2)]}| + |V_{st}(G^{[n(4)]})| + |V_{st}(G^{[n(3)]})| \leq 6p + 2$. \square

Remark 3.8. By following the technique employed in Lemma 3.9, it can be established that given $u_{(\frac{1}{2})}v_i, u_{(\frac{1}{2})}v_{i+2} \in M$ and $u_{(\frac{1}{2})}v_j, u_{(\frac{1}{2})}v_{j+2} \in M$ of G , a $G_{3,m}$ grid, $m \equiv 3 \pmod{4}$, $i + 2 \leq j$, then M is not a MIM of G .

Remark 3.9. Let M be an induced matching of G , a $G_{3,m}$ grid, and i be some fixed positive integer. Suppose $u_{(\frac{1}{2})}v_{i+8(n)} \in M$, for all non-negative integer n for which $1 \leq i + 8(n) \leq m$. Let M be the maximum induced matching of G . Then,

- (a) if $i > 1$, then $i - 1$ is either 2, 3, 4 or 6
- (b) if $i + 8(n) < m$, for the maximum value of n , then $m - (i + 8(n))$ is either 2, 3, 4 or 6.

Based on the results so far, we note that if M is the MIM of G , a $G_{3,m}$ grid, $m \equiv 3 \pmod{4}$, $m \geq 11$, the maximum number of edges of the type $u_{(\frac{1}{2})}v_k$ that is contained in M , k , a positive integer, is $k + 2$ when $m = 8k + 3$ and $k + 3$ when $m = 8k + 7$.

It can be easily established that for H that is a $G_{k,m}$ grid, with $k \equiv 0 \pmod{4}$ and $m \equiv 3 \pmod{4}$, which is induced by U_1, U_2, \dots, U_k , if M_1 is the MIM of H , then, the least saturated vertices in U_k is $\frac{m-1}{2}$. The next result describes the positions of the members of M_1 in $E(H)$ if U_k contains $\frac{m-1}{2}$ saturated vertices.

Lemma 3.10. *Let H be a $G_{k,m}$ grid with $k \equiv 0 \pmod{4}$ and $m \equiv 3 \pmod{4}$ and let U_k contain the least possible, $\frac{m-1}{2}$, saturated vertices for which N remains MIM of H . Then, for any adjacent vertices $v', v'' \in U_k$, edge $v'v'' \notin M$.*

Proof. Induced by $\{U_1, U_2, \dots, U_{k-2}\}$ and $\{U_{k-1}U_k\}$ respectively, let $G_1^{[m]}$ and $G_2^{[m]}$ be partitions of H with $k-2 \equiv 2 \pmod{4}$. It can be seen that $|V_{st}(G_1^{[m]})| = |V_{sb}(G_1^{[m]})| = \frac{km-2m+2}{2}$. Since $|V_{st}H| = \frac{km}{2}$, then $|V_{st}(G_2^{[m]})| \leq m - 1$. Now, let $G_3^{[m]}$ be a $G_{1,m}$ subgrid (a P_m path) of H , induced by U_k . By the hypothesis, U_k contains maximum of $\frac{m-1}{2}$ saturated vertices. Now, let u_kv_i, u_kv_{i+1} be adjacent and saturated vertices of $G_3^{[m]}$. Then there are $\frac{m-5}{2}$ other saturated vertices on $G_3^{[m]}$. Without loss of generality, suppose that each of the remaining $\frac{m-5}{2}$ saturated vertices in $G_3^{[m]}$ is adjacent to some saturated vertex in U_{k-1} . Now, suppose $u_{k-1}v_j$ is a saturable vertex in U_{k-1} and that $v \in V(H)$, such that $u_{k-1}v_jv \in M_1$. Now, $v \notin U_k$, since all the saturable vertices in U_k is saturated. Likewise, suppose $v \in U_{k-1}$ and then $u_{k-1}v_jv \in E(G_4^{[m]})$, where $G_4^{[m]}$ is a $G_{1,m}$ subgrid of H induced by U_{k-1} . Then, clearly, at least one of $u_{k-1}v_j$ and v is adjacent to a saturated vertex in $V_{st}(G_1^{[m]})$. Also, suppose that $v \in U_{k-2}$, since $|V_{sb}(G_1^{[m]})| = |V_{st}(G_1^{[m]})|$, then $|V_{st}(G_1^{[m]})| = |V_{st}(G_1^{[m]} + u_{k-1}u_j)|$. Hence, v is a FSV in G_1^m . Therefore, $|V_{st}H| \leq |V_{st}G_1^{[m]}| + |V_{st}G_2^{[m]}| \leq \frac{km-4}{2}$, which is a contradiction since $|V_{st}(H)| = \frac{km}{2}$, by [7]. □

Remark 3.10. The implication of Lemma 3.10 is that for a grid $H' \subset H$, which is induced by $\{U_1, U_2, \dots, U_{k-2}\} \subset V(H)$, $k-2 \equiv 2 \pmod{4}$, suppose U_k contains the least possible saturated vertices, $\frac{m-1}{2}$, then $u_kv_2, u_kv_4, \dots, u_kv_{m-1}$ are saturated as shown in the example in Figure 3, for which $k = 4$ and $m = 7$.

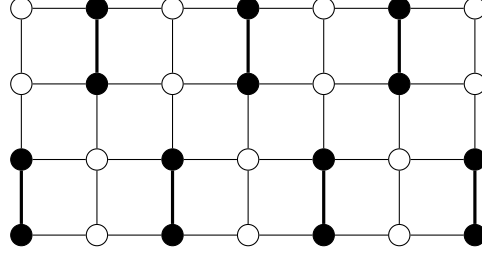


Figure 3. A $G_{4,7}$ Grid with $MIM_G = 11$

Lemma 3.11. Let G be a $G_{3,m}$ with an induced matching M and $G^{(9)}$, induced by $\{V_i, V_{i+2}, \dots, V_{i+8}\}$ be a $G_{3,9}$ subgrid of G . Suppose that $M' \subset M$ is an induced matching of $G^{(9)}$ such that $u_{(\frac{1}{2})}v_i, u_{(\frac{1}{2})}v_{i+8} \in M'$. No other edge $u_{\frac{1}{2}}v_{i+t}, 1 < t < i+7$ is contained in M' . Then for $G'^{(9)} \subset G^{(9)}$, defined as $G'^{(9)} \setminus U_1, |V_{sb}(G'^{(9)})| \leq 8$.

Proof. Let $G^{(7)} = G^{(9)} \setminus \{\{u_1v_{i+1}, u_1v_{i+2}, \dots, u_1v_{i+7}\}, V_i, V_{i+8}\}$. It can be seen that $G^{(7)}$ is a $G_{2,7}$ subgrid of $G^{(9)}$. Clearly also, $G^{(7)} \subset G'^{(9)}$. Since $u_{\frac{1}{2}}v_i, u_{\frac{1}{2}}v_{i+8} \in M'$, then, u_2v_{i+1} and u_2v_{i+7} can not be saturated. Let $G_y \subset G^{(7)}$ be subgraph of $G^{(7)}$, defined as $G^{(7)} \setminus \{u_2v_{i+1}, u_2v_{i+7}\}$. Now, $|V(G_y)| = 12$ and $|V_{sb}(G_y)|$ can be seen to be at most 6. Thus $|V_{sb}(G'^{(9)})| = |V_{sb}(G_y)| + 2 = 8$, since u_2v_i and u_2v_{i+8} are saturated in M' . \square

Remark 3.11. For $U_1 \subset G^{(9)}$ as defined in Lemma 3.11, U_1 contains at least 6 saturated vertices, implying that M' contains two edges whose four vertices are from U_1 .

Corollary 3.12. Let G be a $G_{3,m}$ grid with $m \geq 11$ and $m \equiv 3 \pmod{4}$. If M is a MIM of G . Then M' contains at least $2k'$ edges from U_1 , where $m = 8k' + 3$ or $m = 8k' + 7$.

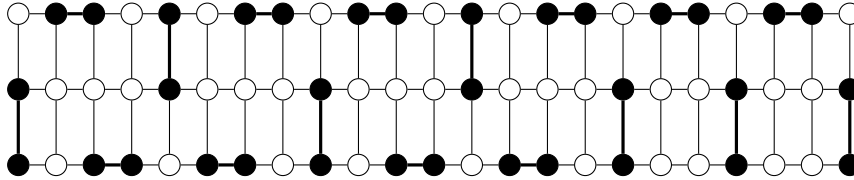
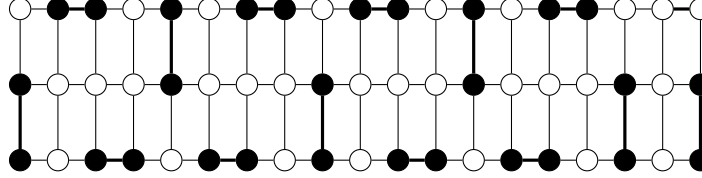


Figure 4. A $G \equiv G_{3,23}$ Grid with $MIM_G = 17$


 Figure 5. A $G \equiv G_{3,14}$ Grid with $MIM_G = 14$

Theorem 3.13. *Let G be a $G_{n,m}$ grid, with $m \geq 23$. Then for $n \equiv 1 \pmod{4}$, $MIM_G \leq \lfloor \frac{2mn-m-3}{8} \rfloor$.*

Proof. For $n \equiv 1 \pmod{4}$, $n-5 \equiv 0 \pmod{4}$. Let G_1 and G_2 be partitions of G with G_1 induced by $\{U_1, U_2, \dots, U_{n-5}\}$ and $\{U_{n-4}, U_{n-3}, U_{n-2}, U_{n-1}, U_n\}$ respectively. Also, let M', M'' be MIM of G_1 and G_2 respectively. Suppose, U_{n-5} contains at least $\frac{m-1}{2}$ saturated vertices, the least U_{n-5} can contain for M' to remain MIM of G_1 . By Theorem 3.7, $U_1 \subset G_2$ (the U_{n-4} of G) contains at least $2k+2$ saturated vertices with $k = \frac{m-3}{4}$. Following the proof of Theorem 3.7, it is shown that M'' contains $\frac{m-3}{4}$ edges of $U_1 \subset G_2$ and either of $u_{(\frac{1}{2})}v_4$ and $u_{(\frac{1}{2})}v_{m-3}$. Now, with $G = G' \cup G''$ and hence, $|M| \leq |M'| + |M''|$, it is obvious therefore, that for each edge $u_\alpha u_\beta \in U_{n-4}$ contained in M'' , either u_α or u_β is adjacent to a saturated vertex on U_{n-5} and also, $u_{n-4}v_4$ (or $u_{n-4}v_{m-3}$) is adjacent to saturated $u_{n-5}v_4$ (or to saturated $u_{n-4}v_{m-3}$). Hence, $|V_{st}(G)| \leq \frac{2mn-m-7}{4}$ and thus, $MIM_G \leq \lfloor \frac{2mn-m-7}{8} \rfloor$. \square

Theorem 3.14. *Let G be a $G_{n,m}$ grid with $n \equiv 3 \pmod{4}$ and $m \equiv 3 \pmod{4}$, $m \geq 11$. Then $MIM_G \leq \lfloor \frac{2mn-m+1}{8} \rfloor$ and $MIM_G \leq \lfloor \frac{2mn-m+5}{8} \rfloor$ for $m = 8k' + 3$ and $m = 8k' + 7$ respectively.*

Proof. The proof follows similar technique as in Theorem 3.13. \square

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¹DEPARTMENT OF COMPUTER AND MATHEMATICAL SCIENCES,
CRAWFORD UNIVERSITY,
NIGERIA
E-mail address: `tayoadebokun@crawforduniversity.edu.ng`

²DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF IBADAN,
IBADAN,
NIGERIA
E-mail address: `olayide.ajayi@mail.ui.edu.ng`; `adelaideajayi@yahoo.com`